This example shows that a minimum of the condition number and RGDOP also can be found when four satellites are located in the vertices of a regular tetrahedron.

Conclusion

In this Note, three measures that can be used in GPS positioning are introduced and detailed. Inequalities describing the relationship between GDOP and condition number are derived, as well as the relationship between GDOP and RGDOP. The results show that GDOP is approximately proportional to the condition number, whereas PDOP is exactly proportional to RGDOP. These results will enhance the understanding of the mathematical aspect of GPS positioning and can be applied to satellite selection and constellation design problems.

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Extension of the Friedland Parameter Estimator to Discrete-Time Systems

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I. Introduction

FRIEDLAND[1] developed a globally convergent nonlinear observer to estimate parameters for continuous-time systems that are affine in the unknown parameter. The derivation of the Friedland observer requires smooth partial derivatives and is, therefore, inherently limited to continuous-time systems. Inasmuch as the realization of the observer will use discrete-time digital sampling, it is desirable to develop a direct discrete-time implementation.

In this Engineering Note, the discrete-time version of Friedland's parameter estimator is derived and extended to the case of time-varying parameters. Estimation of the angular rate of a momentum wheel from quadrature resolver position output is demonstrated. Estimation of the poles of an autoregressive filter is also demonstrated, and the conventional solution is shown to be a subset of this more general technique.

II. Derivation

Consider the discrete-time dynamic system

$$\boldsymbol{x}_{k+1} = f(\boldsymbol{x}_k, \boldsymbol{u}_k, \boldsymbol{p}_k) \tag{1}$$

$$\boldsymbol{p}_{k+1} = A(\boldsymbol{x}_k)\boldsymbol{p}_k \tag{2}$$

where $\boldsymbol{x}$ is the state of the process, $\boldsymbol{u}$ is the control input, $f()$ is the state transition function, $\boldsymbol{p}$ is a vector of parameters, and $A(\boldsymbol{x}_k)$ is the time-dependent (or possibly state-dependent) parameter dynamics matrix. The entire state vector $\boldsymbol{x}_k$ is presumed available for direct measurement.

If the parameter-dependent part of the dynamics are affine in the parameter vector $\boldsymbol{p}$, Eq. (1) can be written

$$\boldsymbol{x}_{k+1} = F(\boldsymbol{x}_k, \boldsymbol{u}_k)\boldsymbol{p}_k + g(\boldsymbol{x}_k, \boldsymbol{u}_k) \tag{3}$$

$$\boldsymbol{p}_{k+1} = A(\boldsymbol{x}_k)\boldsymbol{p}_k \tag{4}$$

The proposed observer for parameter vector $\boldsymbol{p}$ has the form of a reduced-order estimator:

$$\hat{\boldsymbol{p}}_k = A(\boldsymbol{x}_{k-1})\hat{\boldsymbol{p}}_{k-1} + M(\boldsymbol{x}_{k-1})\boldsymbol{x}_k + z_k \tag{5}$$

$$z_{k+1} = -M(\boldsymbol{x}_k)[F(\boldsymbol{x}_k, \boldsymbol{u}_k)\hat{\boldsymbol{p}}_k + g(\boldsymbol{x}_k, \boldsymbol{u}_k)] \tag{6}$$

where $\hat{\boldsymbol{p}}$ is the estimate of $\boldsymbol{p}$, $M(\boldsymbol{x})$ is a state-dependent gain matrix, and z is an intermediate state variable of the same dimension as $\boldsymbol{p}$.

The dynamics of the parameter estimation error $\epsilon_{k+1} = \hat{\boldsymbol{p}}_{k+1} - \boldsymbol{p}_{k+1}$ can be expressed as

$$\epsilon_{k+1} = A(\boldsymbol{x}_k)\hat{\boldsymbol{p}}_k + M(\boldsymbol{x}_k)\boldsymbol{x}_{k+1} + z_{k+1} - A_k\boldsymbol{p}_k \tag{7}$$

$$\begin{aligned}\epsilon_{k+1} &= A(\boldsymbol{x}_k)\epsilon_k + M(\boldsymbol{x}_k)[F(\boldsymbol{x}_k, \boldsymbol{u}_k)\boldsymbol{p}_k + g(\boldsymbol{x}_k, \boldsymbol{u}_k)] \\ &\quad - M(\boldsymbol{x}_k)[F(\boldsymbol{x}_k, \boldsymbol{u}_k)\hat{\boldsymbol{p}}_k + g(\boldsymbol{x}, \boldsymbol{u})]\end{aligned} \tag{8}$$

$$\epsilon_{k+1} = [A(\boldsymbol{x}_k) - M(\boldsymbol{x}_k)F(\boldsymbol{x}_k, \boldsymbol{u}_k)]\epsilon_k \tag{9}$$

which is stable when the eigenvalues of matrix

$$A(\boldsymbol{x}_k) - M(\boldsymbol{x}_k)F(\boldsymbol{x}_k, \boldsymbol{u}_k)$$

are inside the unit circle. The observer design problem is to find a matrix $M(\boldsymbol{x}_k)$ that produces this result.

III. Unmodeled Parameter Dynamics

The effect of an error in the model for the parameter dynamics [Eq. (4)] is to cause a steady-state estimations error. The true plant is given by Eqs. (3) and (4). The observer is constructed as in Eqs. (5) and (6) but with the true $A(\boldsymbol{x}_{k-1})$ replaced by an erroneous $\tilde{A}(\boldsymbol{x}_{k-1})$:

$$\hat{\boldsymbol{p}}_k = \tilde{A}(\boldsymbol{x}_{k-1})\hat{\boldsymbol{p}}_{k-1} + M(\boldsymbol{x}_{k-1})\boldsymbol{x}_k + z_k \tag{10}$$

The resulting error dynamics are given as

$$\epsilon_{k+1} = [\tilde{A}(\boldsymbol{x}_k) - M(\boldsymbol{x}_k)F(\boldsymbol{x}_k, \boldsymbol{u}_k)]\epsilon_k + [\tilde{A}(\boldsymbol{x}_k) - A(\boldsymbol{x}_k)]\boldsymbol{p}_k \tag{11}$$

In steady state, $\epsilon_{k+1} = \epsilon_k$. From Eq. (11), the steady-state error ϵ_∞ is given as

$$\epsilon_\infty = [I - \tilde{A}(\boldsymbol{x}_\infty) + M(\boldsymbol{x}_\infty)F(\boldsymbol{x}_\infty, \boldsymbol{u}_\infty)]^{-1}[\tilde{A}(\boldsymbol{x}_\infty) - A(\boldsymbol{x}_\infty)]\boldsymbol{p} \tag{12}$$

Equation (12) implies that, if the parameter is bounded, then so is the estimation error if $[\tilde{A}(\boldsymbol{x}_k) - M(\boldsymbol{x}_k)F(\boldsymbol{x}_k, \boldsymbol{u}_k)]$ is a stable matrix.

IV. Example 1: Spacecraft Momentum Wheel Rate Estimator

The sine and cosine of angular position of a spacecraft momentum wheel is measured using a resolver with quadrature output. The state is directly measured and is a function of time and wheel velocity:

$$\boldsymbol{x}_k = \begin{bmatrix}\cos\omega kT \\ \sin\omega kT\end{bmatrix} \tag{13}$$

where ω is the wheel angular rate, k is the discrete-time index, and $T = 1$ s is the sample period.

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The discrete-time dynamics are given by

$$x_{k+1} = \begin{bmatrix} \cos\omega(k+1)T \\ \sin\omega(k+1)T \end{bmatrix} \tag{14}$$

$$x_{k+1} = \begin{bmatrix} \cos\omega kT & -\sin\omega kT \\ \sin\omega kT & \cos\omega kT \end{bmatrix} \begin{bmatrix} \cos\omega T \\ \sin\omega T \end{bmatrix} \tag{15}$$

$$x_{k+1} = F(x_k)p \tag{16}$$

$$A_k = \begin{bmatrix} \cos 2\Omega_0 kT & -\sin 2\Omega_0 kT \\ \sin 2\Omega_0 kT & \cos 2\Omega_0 kT \end{bmatrix} \tag{17}$$

where the parameter vector $p = [\cos\omega T \quad \sin\omega T]'$. Parameter vector p will tend to oscillate at twice orbit frequency, $2\Omega_0 = 0.002$ rad/s, and so an oscillatory, time-dependent parameter propagation matrix A_k is modeled as given in Eq. (17).

Remark 1. Matrix $F(x_k)$ is an orthogonal matrix, i.e., $F'F = FF' = I$. This property is useful for the selection of gain matrix M. Because $F'F = I$, then $LF'F = LI$. A gain matrix $M = LF'$ for $0 < L < 2$ guarantees convergence because closed-loop dynamics matrix $I - MF = (1 - L)I$ and the eigenvalues of $(1 - L)I$ are inside the unit circle.

The proposed observer follows Eqs. (5) and (6). Defining

$$M(x) = LF(x_k)' \tag{18}$$

the observer is given

$$\hat{p}_k = A_k\hat{p}_{k-1} + LF(x_{k-1})'x_k + z_k \tag{19}$$

$$z_{k+1} = -LF(x_k)'F(x_k)\hat{p}_k \tag{20}$$

$$z_{k+1} = -L\hat{p}_k \tag{21}$$

The dynamics of the error $\epsilon = \hat{p} - p$ are given by Eq. (9):

$$\epsilon_{k+1} = (1 - L)\epsilon_k \tag{22}$$

Error ϵ is asymptotically stable when $|1 - L| < 1$ or $0 < L < 2$. In this example, gain L was set to 0.05.

The performance of the estimator in which the observer parameter dynamics matrix A_k is matched to the plant model is shown in Fig. 1. The performance of a simplified estimator in which the parameter dynamics matrix is mismatched, i.e., $A_k = I$, is shown in Fig. 2. In the mismatched case, the error is bounded as predicted by the theory.

Remark 2. The convergence rate and observer noise bandwidth is increased by decreasing $|1 - L|$. As with a linear system, there is a tradeoff between observer speed and measurement noise rejection. The effect of gain L on the dynamic response of the observer follows that of linear discrete-time systems and is well described in Ref. 2.

Remark 3. The angular rate ω is calculated by applying the inverse trigonometric function to $\hat{p}$. In the common situation that $2\pi/\omega \ll T$, $\sin\omega T \approx \omega T$ and $\omega \approx \hat{p}_2/T$.

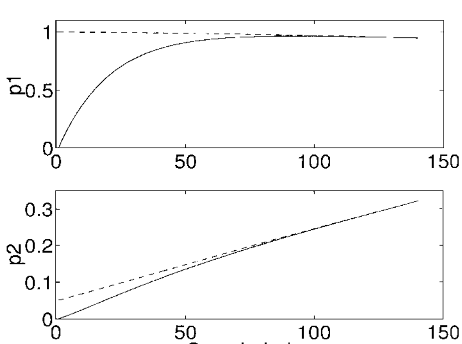

Fig. 1 Convergence of wheel angular displacement estimate for matched observer/plant parameter dynamics (——) to true parameter values (- - -).

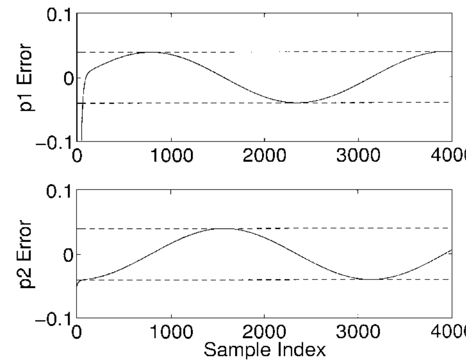

Fig. 2 Wheel angular displacement estimation error for mismatched observer/plant parameter dynamics; estimation error is bounded as predicted by theory.

V. Example 2: Identification of the Poles of an Autoregressive Filter

With slight modification, this parameter estimator can be used to estimate parameters in systems where the n-dimensional state vector is defined as the sequence of n successive outputs. The modification requires inclusion of prior measured states and inputs into the system dynamics. (This formal modification is necessary because, strictly speaking, it is not possible to have full-state measurement on a system in which the forward dynamics depends on states other than the present state. Availability of full-state measurement is the premise underlying both the Friedland observer and the present work.)

Defining $X_k = [x_k, \ldots, x_{k-n+1}]'$, $U_k = [u_k, \ldots, u_{k-n+1}]'$, and modifying Eq. (3), the modified system is represented as

$$x_{k+1} = F(X_k, U_k)p_k + g(X_k, U_k) \tag{23}$$

$$p_{k+1} = A(X_k)p_k \tag{24}$$

and the observer is given

$$\hat{p}_k = A(X_{k-1})\hat{p}_{k-1} + M(X_{k-1})x_k + z_k \tag{25}$$

$$z_{k+1} = -M(X_k)[F(X_k, U_k)p_k + g(X_k, U_k)] \tag{26}$$

With the modified representation, an autoregressive process with nonstationary poles is given as

$$x_{k+1} = F(X_k)p + u_k \tag{27}$$

$$p_{k+1} = A(X_k)p_k \tag{28}$$

where $F(X_k) = X_k'$. The proposed observer follows Eqs. (5) and (6). Define state-dependent gain matrix $M(x)$ for scalar L, $0 < L < 2$, to be

$$M(X_k) = L\frac{X_k}{X_k'X_k} \tag{29}$$

then the observer is given as

$$\hat{p}_k = A(x_{k-1})\hat{p}_{k-1} + L\frac{X_{k-1}}{X_{k-1}'X_{k-1}}x_k + z_k \tag{30}$$

$$z_{k+1} = -L\frac{X_k}{X_k'X_k}[F(X_k)\hat{p}_k + u_k] \tag{31}$$

The parameter error dynamics are given as

$$\epsilon_{k+1} = \left[A(X_k) - L\frac{X_kX_k'}{X_k'X_k}\right]\epsilon_k \tag{32}$$

$$\epsilon_{k+1} = G(X_k)\epsilon_k \tag{33}$$

For $L = 1$ and $A(x_k) = I$, matrix $G(X_k)$ is a rank $n - 1$ idempotent projection matrix with eigenvalues $\lambda \in \{0, 1\}$. The eigenvector

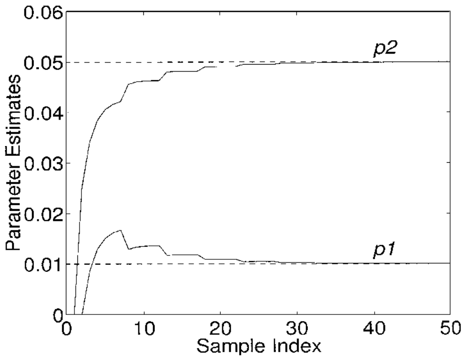

Fig. 3 Convergence of estimates for the pole locations for autoregressive filter (——) to actual pole values (- - -).

corresponding to the stable eigenvalue $\lambda = 0$ is X_k. The eigenvector corresponding to the marginally stable eigenvalue $\lambda = 1$ is in the $(n-1)$-dimension orthogonal subspace $X_k^\perp$. In general, $\lambda \in \{1-L, 1\}$ with eigenvector $e \in \{X_k, X_k^\perp\}$, respectively. Therefore, $0 < L < 2$ guarantees marginal stability. Convergence of this parameter estimator is shown in Fig. 3.

Remark 4. For all L, the eigenvalue corresponding to $X_k^\perp$ remains unity and only marginal stability of the observer can be realized. The entire parameter vector $\boldsymbol{p}$ can be estimated with asymptotically zero error if $X_k \neq 0$ and the direction of X_k varies. This variation is achieved by nonzero input excitation.

Remark 5. The standard parameter estimation projection algorithm, presented in Refs. 3 and 4, is a special case of the new observer presented here. By combining observer equations (30) and (31), the parameter estimation equation is

$$\rho_k = x_k - [F(X_{k-1})\hat{p}_{k-1} + u_{k-1}] \tag{34}$$

$$\hat{p}_k = A(X_{k-1})\hat{p}_{k-1} + L\frac{X_{k-1}}{X'_{k-1}X_{k-1}}\rho_k \tag{35}$$

The parameter identification algorithm from Ref. 3 is

$$\rho_k = x_k - F(X_{k-1})\hat{p}_{k-1} \tag{36}$$

$$\hat{p}_k = A_{k-1}\hat{p}_{k-1} + L\frac{X_{k-1}}{X'_{k-1}X_{k-1}}\rho_k \tag{37}$$

The new observer [Eqs. (34) and (35)] provides for an explicit forcing term and a state-dependent parameter dynamics matrix. (The observer in Ref. 3 can easily accommodate an input forcing function and probably can accommodate a state-dependent parameter dynamics matrix provided that the parameter variation bandwidth is much less than the bandwidth of the measured output.) The observer in Ref. 3 can be viewed as a special case of the new observer for which the full n-dimension state vector in Ref. 3, which is not directly measurable, becomes directly measurable by insertion of n unit delays into the measurement epoch.

VI. Conclusion

A new globally convergent nonlinear parameter estimator was developed. Unlike the conventional regression parameter estimators, this new observer estimates parameters of multistate systems for which all of the system states are simultaneously directly measurable. Because all states are directly, simultaneously measurable, a state-dependent parameter dynamics can be modeled in the new observer.

The first example demonstrated the new observer applied to a multistate system in which each state is simultaneously and directly measurable. The second example demonstrated that the conventional method of parameter estimation for a single-output autoregressive process is a special limited case of this new observer.

Synthesis of the new observer requires finding a satisfactory state-dependent gain matrix $M(\boldsymbol{x}_k)$. Because no general form of $A(\boldsymbol{x}_k)$ or $F(\boldsymbol{x}_k, \boldsymbol{u}_k)$ is assumed and even the dimension of $F(\boldsymbol{x}_k, \boldsymbol{u}_k)$ is not restricted, no general form of $M(\boldsymbol{x}_k)$ is derived. The only stated requirement for selection of gain matrix $M(\boldsymbol{x}_k)$ is that the eigenvalues of $A(\boldsymbol{x}_k) - M(\boldsymbol{x}_k)F(\boldsymbol{x}_k, \boldsymbol{u}_k)$ be inside the unit circle. Experience has shown that, with insight into the system dynamics and a little imagination, a satisfactory $M(\boldsymbol{x}_k)$ can usually be found.

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Continuous Proximate Time-Optimal Control of an Aerodynamically Unstable Rocket

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I. Introduction

TIME-OPTIMAL bang-bang control systems are often impractical because unavoidable measurement noise, disturbances, and nonideal components cause the bang-bang control to switch when the state does not exactly meet the switching criteria. This may degrade tracking performance and produce other bad effects such as a limit cycle about the target state. To reduce or eliminate such undesirable behavior, a number of nonlinear controllers that give near time-optimal response have been developed in the literature.[1–4] McDonald[3] introduced a method called a dual-mode concept, namely, linear mode and nonlinear mode. The dual-mode operation requires the generation of two arbitrary nonlinear functions. One generates the switching curve, and the other determines the boundary of the neighborhood used by the mode selector.

Recently, Workman[5] proposed a controller called proximate time-optimal servomechanism (PTOS) for double-integrator plants. Reference 5 showed that the controller generates solutions that approach the exact minimum-time solutions when the system disturbances and the modeling errors vanish; hence, the term proximate or very near. The controller approximates the switching curve with a strip such that the optimal switching curve is centered along the strip. Unlike the bang-bang controller, the PTOS controller is continuous in the neighborhood of the strip. Near the origin, the PTOS controller switches to a linear feedback law; in this sense, the PTOS controller has dual-mode behavior.

Our approach is motivated by the work of Rauch and Howe.[4] They introduced a controller for second-order systems that attempts to combine the best features of the dual-mode concept[3] with ideas discussed in Ref. 2. The result is a proximate controller, which, unlike the PTOS controller, is a continuous nonlinear function of system state. We have generalized and developed the controller introduced by Rauch and Howe.[4] We, thus, obtain continuous proximate time-optimal (CPTO) controllers, which give near time-optimal response for second-, third-, and higher-order systems.[6] In this Note we present the CPTO controller designed for a second-order unstable plant. The CPTO controller provides nonlinear operation of the servo within a narrow strip in the neighborhood of the switching curve and near linear operation in a neighborhood of the origin,

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